

# VIRASORO CONSTRAINTS IN DRINFELD-SOKOLOV HIERARCHIES

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**ABSTRACT.** We describe a geometric theory of Virasoro constraints in generalized Drinfeld-Sokolov hierarchies. Solutions of Drinfeld-Sokolov hierarchies are succinctly described by giving a principal bundle on a complex curve together with the data of a Higgs field near infinity. String solutions for these hierarchies are defined as points having a big stabilizer under a certain Lie algebra action. We characterize principal bundles coming from string solutions as those possessing connections compatible with the Higgs field near infinity. We show that tau-functions of string solutions satisfy second-order differential equations generalizing the Virasoro constraints of 2d quantum gravity.

## 0. INTRODUCTION

0.1. Drinfeld-Sokolov hierarchies are generalizations of the so-called Korteweg-de Vries (KdV) hierarchy – that is, a family of nonlinear PDEs – to an arbitrary semisimple Lie group  $G$ . For example, the first nontrivial equation of the  $G = SL_2\mathbf{C}$  hierarchy is the well-known KdV equation

$$4u_t = u_{xxx} + 6uu_x$$

for a function  $u(t, x)$ . The whole KdV hierarchy can be conveniently rephrased in terms of the Schrödinger operator  $L = \partial_x^2 + u(x)$  known as the Lax operator in this case. One can similarly obtain the Gelfand-Dikii, or the nKdV hierarchies, by considering an  $n$ -th order differential operator  $L = \partial_x^n + u_{n-2}(x)\partial_x^{n-2} + \dots + u_0(x)$ . Drinfeld and Sokolov's idea was to replace the differential operator  $L$  by a special connection known as a  $G$ -oper. Thus, the phase space of Drinfeld-Sokolov hierarchies is the moduli space of  $G$ -opers on the formal disk, and one recovers the original nKdV phase space when  $G = SL_n\mathbf{C}$ . We will follow a description of *generalized* Drinfeld-Sokolov hierarchies due to Ben-Zvi and Frenkel [BZF1], which starts with a data of a smooth projective curve  $X$ , a semisimple group  $G$  and a maximal torus  $A \subset LG$  of the loop group called a Heisenberg subgroup. The phase space of these hierarchies is the moduli space of affine opers on the formal disk. Considering  $X = \mathbf{CP}^1$  and  $A$  the principal Heisenberg subgroup one recovers the original Drinfeld-Sokolov phase space.

Alternatively, we can also generalize the picture from a different point of view. Using the inverse scattering method one can replace the “coordinate”  $u(x)$  on the phase space of the KdV hierarchy by the spectral properties of the corresponding Schrödinger operator  $L$ . In the situation we are considering here we get a rank 2 vector bundle on  $\mathbf{A}^1$ , whose fiber at  $\lambda \in \mathbf{A}^1$  is the eigenspace  $\text{Ker}(L - \lambda)$ . This vector bundle can be extended to the whole  $\mathbf{CP}^1$  in a canonical way; moreover, the extension carries an extra structure near infinity: the vector bundle comes as a pushforward from the trivial line bundle on a fully-ramified spectral cover  $\text{Spec } \mathbf{C}[[\lambda^{-1/2}]] \rightarrow \text{Spec } \mathbf{C}[[\lambda^{-1}]]$ . This picture has an extension to generalized Drinfeld-Sokolov hierarchies by considering the moduli space (known as the abelianized Grassmannian) of  $G$ -bundles on  $X$  together with a reduction to the Heisenberg subgroup  $A$  near a marked point.

Ben-Zvi and Frenkel defined a natural isomorphism between the oper description of the Drinfeld-Sokolov hierarchy and the spectral description via bundles on curves. The spectral description of integrable hierarchies has an attractive quality that the time evolution is linearized: namely, the Drinfeld-Sokolov flows simply change the data of a reduction near the marked point.

The abelianized Grassmannian is a scheme of infinite type, so to get a handle on this space we would like to be able to describe certain special points. These points also admit an interesting geometric description we are about to describe.

While studying Gelfand-Dikii hierarchies, Krichever considered special Lax operators  $L$ , such that there is another differential operator  $M$ , which commutes with  $L$  and has a coprime order. He called these Lax operators *algebro-geometric*, since they can be recovered from the data of the spectral curve  $\text{Spec } \mathbb{C}[L, M]$  together with a line bundle over it, whose fiber over  $(\lambda, \mu)$  is the joint eigenspace  $\text{Ker}(L - \lambda) \cap \text{Ker}(M - \mu)$ . The reader is referred to [Mu] for a gentle introduction to the classical approach to these solutions.

Instead, we will follow a more general approach to algebro-geometric solutions of generalized Drinfeld-Sokolov hierarchies outlined in [BZF1]. They are described in terms of abstract Higgs bundles of Donagi, such that the cameral cover near the marked point on  $X$  is identified with that of the Heisenberg subgroup  $A$ . The orbits of Drinfeld-Sokolov flows are finite-dimensional and can be identified with torsors over Picard and Prym varieties associated to the cameral covers. Informally, one can say that algebro-geometric solutions admit a sufficiently big stabilizer under the Drinfeld-Sokolov flows. Essentially, the only known solutions of Drinfeld-Sokolov hierarchies are algebro-geometric.

In addition to algebro-geometric solutions we have the so-called *string* solutions, the simplest of which is  $u(x, t) = -\frac{2x}{3t}$  in the KdV case. The corresponding Lax operators  $L$  admit a differential operator  $P$ , such that  $[P, L] = 1$  (the string equation). The famous conjecture of Witten, proved by Kontsevich, states that the partition function of 2d quantum gravity (Gromov-Witten potential of a point) satisfies Virasoro constraints and the equations of the KdV hierarchy. In fact, this partition function is the tau-function associated to the solution written above. As Kac and Schwarz have shown [KS] (see also [AvM]), in the case of nKdV hierarchies, imposing the string equation, the KdV hierarchy, and certain analyticity assumptions is enough to fix the solution uniquely, so that the tau-function will satisfy the Virasoro constraints. The reference [Sch] discusses more general string solutions of nKdV hierarchies.

In this paper we generalize the notion of string solutions from nKdV hierarchies to generalized Drinfeld-Sokolov hierarchies for arbitrary base curves and arbitrary Heisenbergs. Just as algebro-geometric solutions had a sufficiently big stabilizer under the Drinfeld-Sokolov flows, we define a bigger Lie algebra action on the phase space for *varying* curves; the string solutions then possess a big stabilizer under the action of the latter Lie algebra. The goal of the paper is to understand what kind of geometry is behind these solutions and to prove the Virasoro constraints (certain second-order differential equations) on the tau-functions of string solutions.

The plan of the paper is as follows. In [Section 1](#) we review background material on Heisenberg subgroups and abstract Higgs bundles. We then proceed to define the phase space of generalized Drinfeld-Sokolov hierarchies together with Drinfeld-Sokolov flows for a fixed curve  $X$  and in moduli. A geometric definition of tau-functions for generalized Drinfeld-Sokolov hierarchies was missing in the literature, so we give their brief description in [subsection 1.6](#) relating, in particular, to a more algebraic definition in the case  $X = \mathbb{CP}^1$  that can be found in [Wu]. Note,

that in this paper we only consider a linear action of the Virasoro algebra on the tau-function. We do not know a geometric meaning of the non-linear symmetries that appear in *loc. cit.*

In [Section 2](#) we discuss algebro-geometric solutions for generalized Drinfeld-Sokolov hierarchies in a way that will be easily generalized to string solutions. We also discuss the linear differential equations the algebro-geometric tau-functions satisfy.

[Section 3](#) is the main part of the paper. We define string solutions as points having a big stabilizer under the Heisenberg-Virasoro Lie algebra acting on the phase space. We then give a geometric description of string solutions ([Theorem 7](#)) as spectral bundles with a connection, which has a standard structure near the marked point. Generalizing a theorem of F. Plaza Martín [\[PM\]](#) from the case of nKdV hierarchies to generalized Drinfeld-Sokolov hierarchies, we prove that algebro-geometric and string conditions are mutually exclusive if the base curve  $X$  has genus 0 ([Theorem 8](#)). To obtain Virasoro constraints on string tau-functions, we first define the Sugawara embedding for arbitrary Heisenbergs. Using the Sugawara currents we define an action of the negative part of the Virasoro algebra  $\text{Vir}_- = H^0(X \setminus \infty, T_X)$  on the space of tau-functions. When  $X$  has genus 0 and  $G$  is simply-laced, the space of tau-functions is one-dimensional. Moreover, all one-dimensional representations of the negative part of the Virasoro algebra in this case are trivial since  $H^1(\text{Vir}_-, \mathbf{C}) = 0$ , thus we get Virasoro constraints, i.e. second-order differential equations on string tau-functions. This is the content of [Theorem 13](#). We conclude the paper with a description of string solutions in terms of opers.

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## 1. DRINFELD-SOKOLOV HIERARCHIES

### 1.1. Loop groups.

1.1.1. Let  $D = \text{Spec } \mathbf{C}[[z]]$  be the formal disk and  $D^\times = \text{Spec } \mathbf{C}((z))$  the formal punctured disk.

For  $G$  a complex connected reductive group we define the loop group  $LG$  to be the group of maps  $\text{Map}(D^\times, G)$  under pointwise multiplication. It has a natural structure of an ind-scheme. Similarly, we define the positive loop group  $LG_+$  to be the group scheme of loops  $\text{Map}(D, G)$  that extend over the puncture.

1.1.2. *Remark.* It will be useful to think of  $LG$  and  $LG_+$  as group schemes with constant fibers  $G$  over  $D^\times$  and  $D$  respectively.

### 1.2. Heisenberg subgroups.

1.2.1. As all Cartan subgroups of  $G$  are conjugate, the variety of all Cartan subgroups is isomorphic to  $G/N(H)$  for some fixed Cartan subgroup  $H \subset G$ , where  $N(H)$  is the normalizer of  $H$  in  $G$ . It carries a group scheme  $\underline{H}$  whose fiber is the corresponding Cartan subgroup.

**Definition.** A *Heisenberg subgroup*  $A \subset LG$  is a maximal torus in  $LG$  as a group scheme over  $D^\times$ .

In other words, one has a classifying map  $C_A : D^\times \rightarrow G/N(H)$  and the group scheme  $\underline{A}$  is obtained as a pullback  $C_A^* \underline{H}$ .

For  $H \subset G$  a Cartan subgroup we have the homogeneous Heisenberg subgroup  $A = LH$ . The corresponding classifying map  $C_A : D^\times \rightarrow G/N(H)$  is a constant map. In particular, it extends over the puncture. In general, however, the limiting group will not be a Cartan subgroup, but, still, in nice situations it will be a centralizer of a regular element.

The variety  $G/N(H)$ , which parametrizes centralizers of regular semisimple elements, has a compactification  $\widetilde{G/N(H)}$ , which parametrizes centralizers of regular elements. Moreover, it is equipped with a universal abelian group scheme which we also denote by  $\underline{H}$ . The action of  $G$  on  $G/N(H)$  extends to the compactification, so that the group scheme  $\underline{H}$  is  $G$ -equivariant.

**Definition.** A Heisenberg subgroup  $A$  is called *regular* if the corresponding classifying map  $C_A : D^\times \rightarrow G/N(H)$  extends to a map  $C_{A+} : D \rightarrow \widetilde{G/N(H)}$ .

In this case there is an abelian group scheme  $\underline{A}_+ \rightarrow D$  extending  $\underline{A} \rightarrow D^\times$ .

1.2.2. *Higgs bundles and spectral curves.* We refer the reader to [DG] for a comprehensive treatment of abstract Higgs bundles, here we will only sketch the necessary basic facts.

**Definition.** A *regular Higgs bundle*  $(P, \sigma)$  on  $X$  is a  $G$ -torsor  $P \rightarrow X$  together with a  $G$ -equivariant map  $\sigma : P \rightarrow \widetilde{G/N(H)}$ .

For a fixed  $G$ -torsor  $P \rightarrow X$  we call  $\sigma$  a Higgs field.

For example, the data of a regular Heisenberg  $A_+$  is equivalent to a regular Higgs field  $C_{A+}$  on the trivial  $G$ -bundle  $\underline{G} \rightarrow D$ .

There is a natural  $W = N(H)/H$  principal bundle  $G/H \rightarrow G/N(H)$ , which extends to a ramified  $W$ -cover  $\widetilde{G/H} \rightarrow \widetilde{G/N(H)}$ . The pullback of this  $W$ -cover along  $\sigma$  is  $G$ -equivariant and hence descends to a  $W$ -cover of  $X$  known as the *cameral cover*  $X[\sigma]$ . We call a Higgs field  $\sigma$  *unramified* if the image of  $\sigma$  lies inside of  $G/N(H) \subset \widetilde{G/N(H)}$ . In this case the cameral cover is unramified, and so is a  $W$ -torsor.

1.2.3. For  $A$  a regular Heisenberg,  $G$ -torsors induced from  $\underline{A}_+$ -torsors carry an extra structure. Indeed, suppose  $E$  is an  $\underline{A}_+$ -torsor. Then the induced  $G$ -torsor  $P = E \times_{\underline{A}_+} G$  carries a subbundle of  $\text{ad } P$  consisting of regular centralizers. This family gives a section of  $P \times_G \widetilde{G/N(H)} \rightarrow D$ , which is the same as a  $G$ -equivariant map  $P \rightarrow \widetilde{G/N(H)}$ . But this is nothing else but a Higgs field. In fact, we have the following statement:

**Proposition 1** (Donagi-Gaitsgory). *The category of  $\underline{A}_+$ -torsors on  $D$  is equivalent to the category of Higgs bundles  $(P, \sigma)$  together with an isomorphism of  $W$ -covers  $D[P] \xrightarrow{\sim} D[A]$ .*

The Higgs field corresponding to  $A$  is unramified, so the description of  $\underline{A}$ -torsors is more explicit:

**Proposition 2.** *The following three categories are equivalent:*

- (1) *The category of  $\underline{A}$ -torsors on  $D^\times$ .*
- (2) *The category of unramified Higgs fields  $(P, \sigma)$  on  $D^\times$  together with an isomorphism of  $W$ -torsors  $D^\times[P] \xrightarrow{\sim} D^\times[A]$ .*

- (3) *The category of  $N(H)$ -torsors  $P_{N(H)}$  together with an isomorphism of the associated bundle  $P_{N(H)} \times_{N(H)} W \xrightarrow{\sim} D^\times[A]$ .*

1.2.4. Given a Heisenberg  $A$ , we can look at the monodromy of the cameral cover around a loop in  $D^\times$ . It gives a conjugacy class in the Weyl group  $W$ . As Kac and Peterson showed, the  $LG$ -conjugacy classes of Heisenbergs are labeled by the conjugacy class of the monodromy. Roughly, the argument goes as follows. The set of Heisenbergs is isomorphic to the set of maps  $\text{Map}(D^\times, G/N(H))$ . Therefore, the set of  $LG$ -conjugacy classes of Heisenbergs is isomorphic to the set of  $N(H)$ -torsors on  $D^\times$ . The extension of groups

$$1 \rightarrow H \rightarrow N(H) \rightarrow W \rightarrow 1$$

shows that the isomorphism classes of  $N(H)$ -torsors on  $D^\times$  whose induced  $W$ -bundles are isomorphic to a fixed one is a torsor over the group of  $H$ -torsors on  $D^\times$ . But since  $H$  is a torus, the group of  $H$ -torsors is trivial by Hilbert's theorem 90.

1.2.5. Let  $E \rightarrow D^\times$  be an  $\underline{A}$ -torsor. One can view it as a bundle with a varying structure group non-canonically isomorphic to a fixed Cartan  $H$ . To make the isomorphism canonical, consider the variety  $G/H$ , the universal cameral cover of  $G/N(H)$ , which parametrizes Cartan subgroups  $H' \subset G$  together with an isomorphism  $H' \cong H$  given by conjugation. Using these isomorphisms the pullback of the universal group scheme  $\underline{H}$  to the cameral cover is canonically identified with the constant group scheme with fiber  $H$ . Hence, the pullback  $E[A]$  of  $E$  to its own cameral cover  $D^\times[A]$  is naturally an  $H$ -bundle.

Being a pullback to the  $W$ -cover, the underlying space  $E[A]$  is  $W$ -equivariant. However, the  $H$ -torsor structure is only  $N(H)$ -shifted  $W$ -equivariant [DG]. Concretely, this means that

$$w.(hx) = (whw^{-1})(w.x), \quad w \in W, h \in H, x \in E[A].$$

We can identify  $A$  with the  $W$ -equivariant gauge transformations on the cameral cover  $\text{Map}_W(D^\times[A], H)$ .

If  $[s]$  is the conjugacy class of the monodromy in the Weyl group, then the cameral cover  $D^\times[A]$  splits as a disjoint union of  $h : 1$  fully-ramified covers of  $D^\times$ , where  $h$  is the order of  $s$ . In particular, we can construct a Heisenberg from  $s$  in the following way. Let  $t$  be a formal coordinate. Then the action of  $s$  extends from  $\mathfrak{h}$  to  $\mathfrak{h}((t))$  by the following formula:

$$s.(at^m) = (s.a)e^{2\pi im/h}t^m, \quad a \in \mathfrak{h}.$$

The Heisenberg  $\mathfrak{a}$  is then obtained as the subspace  $s$ -fixed vectors in  $\mathfrak{h}((t))$ .

1.2.6. Recall that for  $\pi : P \rightarrow X$  a  $G$ -torsor  $\text{ad } P$  denotes the vector bundle of  $G$ -invariant vertical vector fields on  $P$ , while the Atiyah bundle  $\mathcal{A}_P$  is the bundle of all  $G$ -invariant vector fields. Explicitly,

$$\text{ad } P = (\pi_* T_{P/X})^G, \mathcal{A}_P = (\pi_* T_P)^G.$$

They fit into an exact sequence

$$0 \rightarrow \text{ad } P \rightarrow \mathcal{A}_P \rightarrow T_X \rightarrow 0$$

and a connection by definition is a splitting of this sequence.

By identifying  $\underline{A}$ -torsors with  $N(H)$ -torsors, we can define the bundle of vertical vector fields  $\text{ad } E$  and the Atiyah bundle  $\mathcal{A}_E$  as the bundles corresponding to the  $N(H)$ -torsor. This makes sense since the quotient  $N(H)/H$  is discrete.

We have the Atiyah sequence

$$0 \rightarrow \text{ad } E \rightarrow \mathcal{A}_E \rightarrow T_D \rightarrow 0.$$

It induces a canonical connection  $\nabla_E^{ad}$  on  $\text{ad } E$  in the following way. Let  $v \in T_D$  be a vector field and consider its lift  $\tilde{v} \in \mathcal{A}_E$ . Then one can define

$$\nabla_v^{ad} a := [\tilde{v}, a]$$

for any element  $a \in \text{ad } E$ . This is well-defined since any two lifts  $\tilde{v}$  differ by an element of  $\text{ad } E$ , which is abelian.

One has an identification  $H^0(D^\times, \text{ad } E) \cong \text{Map}_W(D^\times[A], \mathfrak{h}) \cong \mathfrak{a}$ , where  $\mathfrak{a}$  is the Lie algebra of  $A$ . Therefore, we get an action of the Witt algebra  $H^0(D^\times, T_D)$  on the Heisenberg  $\mathfrak{a}$ .

A choice of a coordinate  $z$  on  $D$  gives a vector field  $L_0 = z \frac{\partial}{\partial z} \in T_D$ , which induces a gradation  $\nabla_{L_0}^{ad}$  on  $\mathfrak{a}$ .

**Proposition 3.** *The degree 0 subspace of a Heisenberg  $\mathfrak{a}$  is trivial iff the monodromy is given by a Coxeter element.*

*Proof.* Suppose  $\nabla_{L_0}^{ad} v = 0$  for some  $v \in \mathfrak{a} \cong \text{Map}_W(D^\times[A], \mathfrak{h})$ . The tangent bundle to  $D^\times[A]$  is trivial, so the map  $v$  is annihilated by any vector field.

Therefore,  $v$  is a locally-constant map  $D^\times[A] \rightarrow \mathfrak{h}$ . Moreover, the image is fixed by the monodromy. But Coxeter elements are precisely characterized by the absence of fixed vectors.  $\square$

### 1.2.7. Examples.

- (1) Consider the homogeneous Heisenberg  $A = LH$ . In this case  $\nabla^{ad} = d$  and so the gradation given by  $\nabla_{L_0}^{ad}$  is the homogeneous one. Clearly,  $\mathfrak{h} \subset L\mathfrak{h}$  is the subspace of flat sections and the monodromy is trivial.
- (2) Choose Chevalley generators  $e_1, \dots, e_n, f_1, \dots, f_n, h_1, \dots, h_n$  of  $\mathfrak{g}$ . If  $\theta \in \mathfrak{h}^\vee$  is the maximal root, then  $e_0 = f_\theta \otimes z$ ,  $f_0 = e_\theta \otimes z^{-1}$  and  $\{e_i, f_i, h_i\}_{i=1}^n$  are the Chevalley generators of  $L\mathfrak{g}$ .

Let  $p_{-1} = \sum_{i=0}^n f_i$  be the principal nilpotent element of  $L\mathfrak{g}$ . Then the principal Heisenberg  $A$  by definition is the centralizer of  $p_{-1}$  in  $LG$ .

Let  $h_{Cox}$  be the Coxeter number, i.e. the order of the Coxeter element. The principal gradation on  $L\mathfrak{g}$  is given by assigning degree  $1/h_{Cox}$  to  $e_i$ , degree 0 to  $h_i$  and degree  $-1/h_{Cox}$  to  $f_i$ . For example,  $p_{-1} \in \mathfrak{a}$  has degree  $-1/h_{Cox}$  and multiplication by  $z$  increases the degree by 1.

One has an explicit action of the Witt algebra given by

$$\nabla_{L_0}^{ad} = z \frac{\partial}{\partial z} + \sum_{j=1}^n c_j h_j,$$

where  $c_j$  are solutions to  $\sum_{j=1}^n c_j a_{ij} = 1/h_{Cox}$  for  $\{a_{ij}\}$  the Cartan matrix and  $h_{Cox}$  the Coxeter number.

The monodromy of the cameral cover is a Coxeter element, so the Heisenberg has no degree 0 vectors. Indeed, suppose that  $p \in \mathfrak{a}$  is an element of degree 0. Then



$p = \sum_{j=1}^n c_j h_j$  for some  $c_j \in \mathbf{C}$ . Since  $p$  centralizes  $p_{-1}$ ,

$$- \sum_{i=1, j=1}^n c_j a_{ij} f_i + \sum_{j=1}^n c_j \theta(h_j) e_\theta = 0.$$

Since  $\mathfrak{g}$  is semisimple, the Cartan matrix  $a_{ij}$  is nondegenerate. So,  $c_j = 0$ , i.e.  $p = 0$ .

**1.3. Grassmannians.** In this section we define the phase space of the generalized Drinfeld-Sokolov hierarchy together with a collection of vector fields, which make it into an integrable system.

1.3.1. The input data for a generalized Drinfeld-Sokolov hierarchy consists of:

- A smooth projective curve  $X$ .
- A semisimple simply-connected group  $G$ .
- A choice of a Heisenberg subgroup  $A \subset LG$ .

The Drinfeld-Sokolov Grassmannian  $\widehat{\text{Gr}}_g$  is the moduli stack whose  $S$ -points parametrize the following data:

- A family of genus  $g$  smooth projective curves  $X \rightarrow S$  together with a marked point  $\infty : S \rightarrow X$ . We denote by  $X_0 = X \setminus \infty$  the affine part and  $X_\infty$  the spectrum of the completed local ring at infinity.
- A local coordinate  $z : X_\infty \xrightarrow{\sim} D \times S$ .
- A  $G$ -torsor  $P \rightarrow X$  together with a reduction of  $P|_{X_\infty}$  to an  $z^* \underline{A}_+$ -torsor  $E \rightarrow X_\infty$ .
- A trivialization  $E \xrightarrow{\sim} z^* \underline{A}_+$ .

To not obscure the notation, the local coordinate  $z$  will be implicit from now on.

Note, that a trivialization of  $E$  induces a trivialization of  $P|_{X_\infty}$ , hence the Grassmannian  $\widehat{\text{Gr}}_g$  is independent of the choice of a Heisenberg.

Let  $\text{Gr}_g$  be the moduli space obtained from  $\widehat{\text{Gr}}_g$  by forgetting the trivialization of  $E$ . We have the maps

$$\widehat{\text{Gr}}_g \rightarrow \text{Gr}_g \rightarrow \widehat{\mathcal{M}}_{g,1},$$

where  $\widehat{\mathcal{M}}_{g,1}$  is the moduli space of genus  $g$  curves with a marked point together with a choice of a local coordinate. We denote by  $\widehat{\text{Gr}}_X$  and  $\text{Gr}_X$  the fibers of  $\widehat{\text{Gr}}_g \rightarrow \widehat{\mathcal{M}}_{g,1}$  and  $\text{Gr}_g \rightarrow \widehat{\mathcal{M}}_{g,1}$  respectively over a curve  $X \in \widehat{\mathcal{M}}_{g,1}$ . The space  $\text{Gr}_X$  is known as the *abelianized Grassmannian*.

One can explicitly realize  $\widehat{\text{Gr}}_X$  and  $\text{Gr}_X$  as the spaces of cosets

$$\widehat{\text{Gr}}_X = LG_- \backslash LG, \quad \text{Gr}_X = LG_- \backslash LG / A_+,$$

where  $LG_- = \text{Map}(X_0, G)$  is the ind-group of loops that extend away from the marked point.

1.3.2. *Remark.* If two Heisenbergs  $A$  and  $A'$  are  $LG_+$ -conjugate, the corresponding Drinfeld-Sokolov phase spaces are canonically isomorphic and the isomorphism intertwines the flows. However, as noted in [BZF1], there are continuous families of  $LG_+$ -conjugacy classes of Heisenbergs in the same  $LG$ -conjugacy class.

1.3.3. *Examples.* All known examples of Drinfeld-Sokolov hierarchies start with a genus 0 curve, so let  $X = \mathbf{CP}^1$ . In this case we have an open dense subset  $\mathrm{Gr}_X^{\text{big cell}} \subset \mathrm{Gr}_X$  consisting of trivializable  $G$ -torsors  $P$ .

We have the following standard choices of Heisenbergs:

- (1)  $A = LH$  is the homogeneous Heisenberg. For  $G = SL_2\mathbf{C}$  one gets the non-linear Schrödinger hierarchy.
- (2)  $A$  is the principal Heisenberg. Then the big cell  $\mathrm{Gr}_X^{\text{big cell}}$  parametrizes  $G$ -opers on the disk and we recover the original description of Drinfeld and Sokolov. For example, the case  $G = SL_2\mathbf{C}$  corresponds to the KdV hierarchy and  $G = SL_n\mathbf{C}$  to its generalizations known as Gelfand-Dikii or nKdV hierarchies.

The reader is referred to [Kr] for explicit coordinates on Drinfeld-Sokolov phase spaces.

#### 1.4. Line bundles.

1.4.1. Let  $\langle, \rangle$  be an Ad-invariant bilinear form on  $\mathfrak{g}$  taking even values on the coroots. It defines a central extension

$$1 \rightarrow \mathbf{C}^\times \rightarrow \widehat{LG} \rightarrow LG \rightarrow 1.$$

On the level of Lie algebras, one has  $\widehat{L\mathfrak{g}} \cong L\mathfrak{g} \oplus \mathbf{C}$  as vector spaces together with the Lie bracket

$$[(a, \alpha), (b, \beta)] = [a, b] + \mathrm{Res}_{z=0} \langle a, db \rangle.$$

In particular, we see that the central extension is trivial when restricted to  $LG_-$ , i.e. we have an embedding  $LG_- \subset \widehat{LG}$ . Therefore, we get a line bundle

$$LG_- \backslash \widehat{LG} \rightarrow LG_- \backslash LG.$$

Since the central extension restricted to  $LG_+$  is also split, thus defined line bundle is  $LG_+$ -equivariant, i.e. it descends to a line bundle  $\mathcal{L}$  on the moduli space of bundles  $\mathrm{Bun}_G(X) \cong LG_- \backslash LG / LG_+$ . Similarly, it descends to the abelianized Grassmannian  $\mathrm{Gr}_X$ .

1.4.2. So far we have defined line bundles on  $\mathrm{Gr}_X$ , and the question arises whether they extend to the whole Grassmannian  $\mathrm{Gr}_g$ . We have a section  $\widehat{\mathcal{M}}_{g,1} \rightarrow \mathrm{Gr}_g$  given by considering the trivial bundle, which gives a splitting of the pullback map  $\mathrm{Pic} \widehat{\mathcal{M}}_{g,1} \rightarrow \mathrm{Pic} \mathrm{Gr}_g$ .

Pick a curve  $X \in \widehat{\mathcal{M}}_{g,1}$ . Then we have an exact sequence

$$1 \rightarrow \mathrm{Pic} \widehat{\mathcal{M}}_{g,1} \rightarrow \mathrm{Pic} \mathrm{Gr}_g \rightarrow \mathrm{Pic} \mathrm{Gr}_X,$$

where the last map is the restriction to the fiber over  $X$ . Laszlo [La] has shown that one can carry out a construction of the line bundle  $\mathcal{L}$  in families thus obtaining a splitting  $\mathrm{Pic} \mathrm{Gr}_X \rightarrow \mathrm{Pic} \mathrm{Gr}_g$ , which produces a split exact sequence

$$1 \rightarrow \mathrm{Pic} \widehat{\mathcal{M}}_{g,1} \rightarrow \mathrm{Pic} \mathrm{Gr}_g \rightarrow \mathrm{Pic} \mathrm{Gr}_X \rightarrow 1.$$

1.4.3. *Remark.* In types A and C the splitting  $\mathrm{Pic} \mathrm{Gr}_X \rightarrow \mathrm{Pic} \mathrm{Gr}_g$  can be obtained as the determinant line bundle of a fundamental representation.

#### 1.5. Lie algebra actions.



1.5.1. Recall the definition of the Atiyah bundle  $\mathcal{A}_E$  of an  $\underline{A}$ -torsor  $E \rightarrow D^\times$ . The Atiyah sequence

$$0 \rightarrow \text{ad } E \rightarrow \mathcal{A}_E \rightarrow T_D \rightarrow 0$$

is non-canonically split, so we have an identification  $\mathcal{A} := H^0(D^\times, \mathcal{A}_E) \cong \text{Vir} \ltimes \mathfrak{a}$ .

There is an action of the Lie algebra  $\mathcal{A}$  on the Grassmannian  $\widehat{\text{Gr}}_g$  given by deforming both the curve  $X$  and the bundle  $E$ . Note, that the action of  $\mathcal{A}$  does not descend to the Grassmannian  $\text{Gr}_g$  since  $\mathcal{A}$  and  $\mathfrak{a}_+$  do not commute.

The subalgebra  $\mathcal{A}^0 = H^0(D^\times, \text{ad } E) \cong \mathfrak{a}$  of vertical vector fields acts along the fibers of  $\widehat{\text{Gr}}_g \rightarrow \widehat{\mathcal{M}}_{g,1}$ , i.e. it preserves  $\widehat{\text{Gr}}_X$ . Moreover, the action descends to the abelianized Grassmannian  $\text{Gr}_X$ .

This action of  $\mathcal{A}^0$  generates the Drinfeld-Sokolov flows, while  $\mathcal{A}$  represents the so-called Orlov-Shulman extended symmetries.

1.5.2. As line bundles  $\mathcal{L}$  on  $\widehat{\text{Gr}}_g$  correspond to central extensions of  $LG$ , the action of  $\mathcal{A}^0 \cong \mathfrak{a}$  lifts canonically to the line bundle by restricting the central extension to the Heisenberg. Similarly, we have a central extension  $\widehat{\mathcal{A}}$ , which lifts the action of  $\mathcal{A}$  to the line bundle.

## 1.6. Tau-function.

1.6.1. The Lie algebra  $\mathcal{A}$  exponentiates to an ind-group  $K = \text{Aut}(D^\times) \ltimes A$ . Similarly, there is a central extension  $\widehat{K}$  which lifts the action of  $K$  to the line bundle  $\mathcal{L}$  on  $\widehat{\text{Gr}}_g$ .

Consider the action and projection maps

$$\widehat{\text{Gr}}_g \times \widehat{K} \begin{matrix} \xrightarrow{a} \\ \xrightarrow{p} \end{matrix} \widehat{\text{Gr}}_g.$$

Then the action of  $\widehat{K}$  on  $\mathcal{L}$  can be expressed as the data of an isomorphism  $p^*\mathcal{L} \cong a^*\mathcal{L}$ .

Fix a nonzero section  $\sigma \in H^0(\text{Gr}_g, \mathcal{L}) \cong H^0(\widehat{\text{Gr}}_g, \mathcal{L})^{A+}$ . Then one defines a rational function on  $\widehat{\text{Gr}}_g \times \widehat{K}$  called the *extended tau-function*  $\bar{\tau}$  as

$$\bar{\tau} = \frac{p^*\sigma}{a^*\sigma}.$$

Explicitly, one has

$$\bar{\tau}(P, g) = \frac{\sigma(g^{-1}P)}{g^{-1}\sigma(P)}.$$

Thus, it measures the failure of the section  $\sigma$  to be  $\widehat{K}$ -invariant. Since  $\sigma$  is  $A_+$ -invariant, the extended tau-function is a function on  $\widehat{\text{Gr}}_g \times \widehat{K}/A_+$ .

Fixing a point  $P \in \widehat{\text{Gr}}_g$  we define the *tau-function* of  $P$  as the restriction of the extended tau-function  $\bar{\tau}$  to the slice  $\{P\} \times \widehat{A}/A_+$ . It is a rational function on  $\widehat{A}/A_+$ .

1.6.2. Although  $A$  is abelian, its central extension  $\widehat{A}$  is not. Its failure is measured by the commutator pairing  $c : A \times A \rightarrow \mathbf{C}^\times$  given by arbitrarily lifting the elements of  $A$  to the central extension and computing the commutator.

For example, suppose  $a \in A_+$ . Then

$$\tau_{aP}(g) = \frac{\sigma(g^{-1}aP)}{g^{-1}\sigma(aP)} = \frac{a\sigma(g^{-1}P)}{g^{-1}a\sigma(P)} = c(a, g^{-1}) \frac{a\sigma(g^{-1}P)}{ag^{-1}\sigma(P)} = c(a, g^{-1})\tau_P(g).$$

In particular, the tau-function is not well-defined as a function on  $\text{Gr}_g$  and is instead a section of an associated line bundle to  $\widehat{\text{Gr}}_g \rightarrow \text{Gr}_g$ .

1.6.3. There is an equivalent, more algebraic, way to write down the tau-function. Fix a point  $P \in \widehat{\text{Gr}}_X$  and pull back the line bundle  $\mathcal{L}$  to the orbit  $\widehat{A}$  of the Drinfeld-Sokolov flows.  $\sigma$  defines a section of  $\mathcal{L}$ ; in particular, it gives a connection  $d \log \sigma$  with regular singularities at the zeros of  $\sigma$ .

We define a meromorphic 1-form  $d \log \tau$  on  $\widehat{A}$  by

$$d \log \tau[a] = d \log \sigma[a] - v_a,$$

where  $a \in \widehat{\mathfrak{a}}$  and  $v_a \in \mathcal{A}_{\mathcal{L}}$  is the vector field exhibiting the action of  $\widehat{A}$  on  $\mathcal{L}$ .

The connection  $d \log \sigma$  splits the sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{A}_{\mathcal{L}} \rightarrow T_{\widehat{A}} \rightarrow 0$$

whenever  $\sigma$  does not vanish, which happens generically when  $X$  has genus 0. Then  $d \log \tau[a]$  is simply the image of  $-v_a$  under the splitting  $\mathcal{A}_{\mathcal{L}} \rightarrow \mathcal{O}$ . This is the definition of tau-functions, which can be found in [Wu].

## 2. ALGEBRO-GEOMETRIC SOLUTIONS

### 2.1. Geometric description.

2.1.1. Recall that the Grassmannian  $\widehat{\text{Gr}}_X$  is isomorphic to  $LG_- \backslash LG$ , where the corresponding element of  $LG$  is the transition function from the formal disk near infinity to the affine part. Using the trivialization of the torsor  $E$  on the formal disk, we can identify  $H^0(D^\times, \text{Ad } P) \cong LG$  and  $H^0(D^\times, \text{Ad } E) \cong A$ .

**Lemma 4.** *The stabilizer of the  $LG$ -action on  $\widehat{\text{Gr}}_X$  is  $H^0(X_0, \text{Ad } P) \subset H^0(D^\times, \text{Ad } P)$ .*

*Proof.* Suppose an element  $g \in LG$  fixes a coset  $[\gamma] \in LG_- \backslash LG$ . Explicitly, this means that there is an element  $\gamma \in LG_-$ , such that  $\gamma g = \gamma_- \gamma$  or, equivalently,  $g = \gamma^{-1} \gamma_- \gamma$ . But  $\gamma$  is the transition function from the formal disk to the affine part, so we get that  $g \in H^0(X_0, \text{Ad } P)$ .  $\square$

**Corollary 5.** *The stabilizer of the  $\mathfrak{a}$ -action on  $\widehat{\text{Gr}}_X$  is  $\mathcal{A}_{stab}^0 = \mathfrak{a} \cap H^0(X_0, \text{ad } P)$ .*

2.1.2. From this description we see that  $\mathcal{A}_{stab}^0$  has a natural structure of an  $\mathcal{O}(X_0)$ -module, so we can localize it to get a vector bundle  $c$  on  $X_0$ :

$$c = \mathcal{A}_{stab}^0 \otimes_{\mathcal{O}(X_0)} \mathcal{O}_{X_0}.$$

By construction, the restriction  $c|_{D^\times}$  is a subbundle of  $\text{ad } E$ . Therefore, the rank of  $c$  is bounded from above by the rank of  $\text{ad } E$ , which coincides with  $\text{rk } G$ , the rank of the group  $G$ .

**Definition.** A point  $P \in \widehat{\text{Gr}}_X$  is *algebra-geometric* if the rank of  $c$  coincides with  $\text{rk } G$ .

Clearly, this condition is independent of the trivialization of  $E$ , so it makes sense to consider algebra-geometric points of  $\text{Gr}_X$ .

2.1.3. In the algebro-geometric case we have two subbundles  $c \subset \text{ad } P|_{X_0}$  and  $\text{ad } E \subset \text{ad } P|_D$ , which are identified on the intersection. Therefore, we can glue them together to get a bundle  $c$  on the whole  $X$ .  $c$  has rank  $\text{rk } G$  and is a subbundle of abelian Lie algebras, hence it defines an extension of a Higgs field  $\text{ad } E \subset \text{ad } P|_D$  to a regular Higgs field on  $X$ . The converse is also true:

**Proposition 6.** *A point  $P \in \text{Gr}_X$  is algebro-geometric iff the subbundle  $\text{ad } E \subset \text{ad } P|_D$  over  $D$  extends to a regular Higgs field on  $P \rightarrow X$ .*

*Proof.* Suppose that  $c'$  is the extension of  $\text{ad } E$  to  $X$ . Then

$$H^0(X_0, c') \subset \mathfrak{a} \cap H^0(X_0, \text{ad } P) \cong \mathcal{A}_{stab}^0.$$

Therefore,  $c'|_{X_0} \subset c$ . Thus, the rank of  $c$  is bounded from below by  $\text{rk } G$  and hence coincides with  $\text{rk } G$ , i.e. the point is algebro-geometric.  $\square$

We see that algebro-geometric points in  $\text{Gr}_X$  can be reconstructed from the following data:

- A  $G$ -torsor  $P \rightarrow X$ .
- A regular Higgs field  $c \subset \text{ad } P$ .
- An isomorphism of cameral covers  $D[c|_D] \cong D[A]$ .

2.1.4. Consider two points  $P_1, P_2 \in \text{Gr}_X$  which differ by an action of  $A$ . The corresponding Higgs fields  $c_{P_1}$  and  $c_{P_2}$  are isomorphic away from infinity, so we have an isomorphism of cameral covers  $X_0[c_{P_1}] \cong X_0[c_{P_2}]$ . Since both  $X[c_{P_1}]$  and  $X[c_{P_2}]$  are projective curves, this gives an isomorphism  $X[c_{P_1}] \cong X[c_{P_2}]$ . Using the results of [DG] we see that an algebro-geometric orbit of  $A$  is a torsor over the corresponding Prym variety. In particular, it is finite-dimensional.

## 2.2. Tau-function.

2.2.1. Consider the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \mathcal{A}_{stab}^0 & \longrightarrow & \mathcal{A}^0 & \longrightarrow & T_{\text{Gr}_X} \longrightarrow 0 \\
 & & \nearrow & & \uparrow & & \uparrow \\
 & & \hat{\mathcal{A}}^0 & \longrightarrow & \mathcal{A}_{\mathcal{L}} & & \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{O}_{\text{Gr}_X} & = & \mathcal{O}_{\text{Gr}_X} & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

where the middle column is a pullback of the rightmost column along  $\mathcal{A}^0 \rightarrow T_{\text{Gr}_X}$ .

There is a unique lift  $\mathcal{A}_{stab}^0 \rightarrow \hat{\mathcal{A}}^0$ , such that the composite  $\mathcal{A}_{stab}^0 \rightarrow \hat{\mathcal{A}}^0 \rightarrow \mathcal{A}_{\mathcal{L}}$  is zero.

Let  $v \in \mathcal{A}_{stab}^0(P)$  be any vector stabilizing a point  $P \in \text{Gr}_X$ . Since  $\mathcal{A}_{stab}^0 \rightarrow \mathcal{A}_{\mathcal{L}}$  is the zero map,  $v$  preserves the fiber of  $\mathcal{L}$  at  $P$ . Therefore, the tau-function

$$\tau_P(g) = \frac{\sigma(g^{-1}P)}{g^{-1}\sigma(P)}$$

obeys a first-order differential equation

$$v\tau_P = 0$$

for any  $v \in \mathcal{A}_{stab}^0(P)$ . This simply means that the tau-function descends to a well-defined function on the orbit.

### 3. STRING SOLUTIONS

#### 3.1. Geometric description.

3.1.1. Algebro-geometric points were characterized by the property that enough elements of the Heisenberg  $\mathfrak{a}$  stabilize a point. Similarly, one can consider the action of the Virasoro-Heisenberg algebra  $\mathcal{A}$ , which can be thought of as the algebra of first-order differential operators with coefficients in  $\mathfrak{a}$ .

Just as  $\mathcal{A}_{stab}^0 = \mathcal{A}^0 \cap H^0(X_0, \text{ad } P)$ , we also have  $\mathcal{A}^{stab} = \mathcal{A} \cap H^0(X_0, \mathcal{A}_P)$ . In fact, we have the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X_0, \text{ad } P) & \longrightarrow & H^0(X_0, \mathcal{A}_P) & \longrightarrow & H^0(X_0, T_X) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & \mathcal{A}_{stab}^0 & \longrightarrow & \mathcal{A}_{stab} & \longrightarrow & H^0(X_0, T_X) \end{array}$$

Note, that there is no reason to expect in general that the map  $\mathcal{A}_{stab} \rightarrow H^0(X_0, T_X)$  is surjective.

**Definition.** A point  $P \in \widehat{\text{Gr}}_X$  is *string* if the map  $\mathcal{A}_{stab} \rightarrow H^0(X_0, T_X)$  is surjective.

If  $X$  has genus 0,  $A$  is the principal Heisenberg and  $G = SL_n \mathbf{C}$ , Schwarz [Sch] showed that this definition is equivalent to the so-called string equation, hence the name. We will return to the issue of the string equation in [subsection 3.3](#).

Note, that in the string case the sequence of Lie algebras

$$0 \rightarrow \mathcal{A}_{stab}^0 \rightarrow \mathcal{A}_{stab} \rightarrow H^0(X_0, T_X) \rightarrow 0$$

is non-canonically split since  $X$  has dimension 1 and  $T_X$  is locally-free.

3.1.2. Let us pick a splitting  $H^0(X_0, T_X) \rightarrow \mathcal{A}_{stab}^0$  and consider the composite

$$H^0(X_0, T_X) \rightarrow \mathcal{A}_{stab} \hookrightarrow H^0(X_0, \mathcal{A}_P).$$

By definition, this means that  $P$  carries a connection on the affine part  $X_0$ . Moreover, the connection on  $D^\times$  preserves the Heisenberg in the sense that for any vectors  $v \in T_{D^\times}$  and  $a \in \mathfrak{a}$  we have  $[\nabla_v, a] \in \mathfrak{a}$ . We get the following statement:

**Theorem 7.** *A point  $P \in \widehat{\text{Gr}}_X$  is string iff the torsor  $P$  has a connection  $\nabla$  on the affine part  $X_0$ , such that  $\nabla_{z\partial/\partial z}$  induces the canonical gradation on the Heisenberg  $\mathfrak{a}$ .*

*Proof.* Suppose that we have a connection  $\nabla$  and  $\nabla_{z\partial/\partial z}$  induces the canonical gradation. Since the module of vector fields is free over  $\mathcal{O}(D^\times)$ , we see that the whole action of the Witt algebra on the Heisenberg  $\mathfrak{a}$  coincides with the canonical action. We want to prove that it implies that the image of  $\nabla$  lands  $H^0(D^\times, \mathcal{A}_E) \subset H^0(D^\times, \mathcal{A}_P)$ .

Pick a splitting  $s : H^0(D^\times, T_X) \rightarrow H^0(D^\times, \mathcal{A}_E)$ . Since  $\nabla$  gives the canonical action of the Witt algebra on the Heisenberg, the difference  $\nabla_v - s_v$  is an element  $g(v)$  of  $H^0(D^\times, \text{ad } P)$  for any  $v \in H^0(D^\times, T_X)$ . Moreover, as both  $\nabla$  and  $s$  preserve the Heisenberg,  $g(v)$  commutes with  $\mathfrak{a}$ . But then  $g(v) \in \mathfrak{a}$  and hence  $\nabla$  lands in  $H^0(D^\times, \mathcal{A}_E)$ .

We see that there is a map  $H^0(X_0, T_X) \rightarrow H^0(D^\times, \mathcal{A}_E) \cap H^0(X_0, \mathcal{A}_P)$  and hence the point is string.  $\square$

We get the following geometric structure on string solutions:

- A  $G$ -torsor  $P \rightarrow X$  together with an  $\underline{A}_+$ -reduction  $E \rightarrow D$ .
- A subbundle  $c \subset \text{ad } P|_{X_0}$  of abelian Lie algebras, such that  $c|_{D^\times} \subset \text{ad } E|_{D^\times}$ .
- A connection  $\nabla$  on the affine part  $X_0$  which preserves  $c$  and gives the canonical gradation on the Heisenberg  $H^0(D^\times, \text{ad } E)$ .

As before,  $c$  is the localization of  $\mathcal{A}_{stab}^0$ .

3.1.3. If the genus of  $X$  is zero, we can try to exploit the action of  $\mathfrak{sl}_2(\mathbf{C}) \cong H^0(X, T_X)$  on  $c$ .

**Theorem 8.** *Suppose  $X$  has genus 0 and the Heisenberg  $A$  is LG-conjugate to the principal one. Then string points have  $c = 0$ .*

*Proof.* Let  $v \in H^0(X, T_X)$  be a regular vector field with a zero of order 2 at  $\infty$ . In local coordinates  $v = z^2 \frac{\partial}{\partial z}$ . The corresponding derivation of  $H^0(D^\times, \mathcal{O}_X)$  has order 1. Therefore,  $\nabla_v$  also has order 1 on the Heisenberg  $\mathfrak{a}$ .

Take a nonzero element  $e \in \mathcal{A}_{stab}^0$ .  $\nabla_v$  raises its order, so for some  $n$  we have  $(\nabla_v)^n e \in H^0(D, \text{ad } E)$ . On the other hand,

$$H^0(D, \text{ad } E) \cap H^0(X_0, \text{ad } P) \subset H^0(X, \text{ad } P)$$

is finite-dimensional, so, possibly increasing  $n$ , we can assume that  $(\nabla_v)^n e = 0$ . Let  $n$  be the minimal such exponent, so that  $s = (\nabla_v)^{n-1} e \neq 0$ . Then  $s$  is annihilated by  $\nabla$ , i.e. it is a flat section. But it cannot happen by assumption on the Heisenberg and **Proposition 3**. Therefore,  $\mathcal{A}_{stab}^0 = 0$ .  $\square$

In particular, since  $c = 0$ , we see that string points cannot be algebro-geometric.

A similar statement was obtained previously by F. Plaza Martín [PM] when  $G = SL_n \mathbf{C}$  and  $A$  is the principal Heisenberg.

3.2. **Virasoro constraints.** Any vector  $a \in \mathcal{A}_{stab}$  annihilates the extended tau-function

$$a\overline{\tau} = 0$$

since it preserves the section  $\sigma$ . However, unless  $a \in \mathcal{A}_{stab}^0$ , such a constraint does not make sense on the ordinary tau-function since it has components in the Virasoro direction.

To get rid of these components, we would like to know how the tau-function changes under the Virasoro action.

3.2.1. *Sugawara construction for twisted Heisenbergs.* In this section we define an embedding of the Virasoro algebra into the universal enveloping algebra of a Heisenberg called the Sugawara construction, e.g. see [BZF2].

Recall from subsection 1.2 that a Heisenberg  $\mathfrak{a}$  can be obtained as the subspace of  $s$ -invariants in  $\mathfrak{h}((t))$ , where  $s$  acts on  $\mathfrak{h}((t))$  by

$$s.(at^m) = (s.a)e^{2\pi im/h}t^m, \quad a \in \mathfrak{h}.$$

Similarly, one can obtain the central extension  $\widehat{\mathfrak{a}}$  as the subspace of  $s$ -invariants in  $\widehat{\mathfrak{h}((t))}$ , where  $s$  acts trivially on the central element.

Consider a decomposition

$$\mathfrak{h} = \bigoplus_{m=0}^{h-1} \mathfrak{h}_m$$

into eigenspaces of  $s$ , where  $s$  acts on  $\mathfrak{h}_m$  by  $\exp(2\pi im/h)$ . We denote the dimension  $d_\alpha = \dim \mathfrak{h}_\alpha \bmod h$ .

Since  $s$  is orthogonal with respect to the inner product  $\langle, \rangle$  on  $\mathfrak{h}$ , the spaces  $\mathfrak{h}_m$  and  $\mathfrak{h}_{h-m}$  are naturally paired. Let  $\{a_m^i\}_i$  be a basis of  $t^m \mathfrak{h}_m$  and let  $\{a_{-m}^i\}_i$  be the dual basis of  $t^{-m} \mathfrak{h}_{h-m}$ .

Let us define the universal enveloping algebra  $U_k(\widehat{\mathfrak{a}})$  as the quotient  $U(\widehat{\mathfrak{a}})/(\mathbf{K} - k \cdot 1)$ , where  $\mathbf{K}$  is the central element of  $\widehat{\mathfrak{a}}$ . We can similarly define the universal enveloping algebras  $U_c(\text{Vir})$  and  $U_{k,c}(\text{Vir} \ltimes \widehat{\mathfrak{a}})$ .

The Sugawara currents we are about to define will involve infinite expressions in the elements  $a_\alpha^i$ , so they belong to a completion of the universal enveloping algebra  $U_k(\widehat{\mathfrak{a}})$  in the “positive” direction:

$$\widehat{U}_k(\widehat{\mathfrak{a}}) = \varprojlim_n U_k(\widehat{\mathfrak{a}})/(\widehat{\mathfrak{a}}_n U_k(\widehat{\mathfrak{a}})),$$

where we denote  $\widehat{\mathfrak{a}}_n = t^{hn} \widehat{\mathfrak{a}}_+$ . Similarly, we can define a completion  $\widehat{U}_{k,c}(\text{Vir} \ltimes \widehat{\mathfrak{a}})$ .

**Proposition 9.** *The elements of  $\widehat{U}_k(\widehat{\mathfrak{a}})$*

$$L_n^S = \frac{1}{kh} \sum_{\alpha < nh/2, i} a_{hn-\alpha}^i a_\alpha^{\bar{i}} + \frac{1}{2kh} \sum_i a_{hn/2}^i a_{hn/2}^{\bar{i}} - \delta_{n,0} \frac{1}{4h^2} \sum_{0 < l < h} d_l l(h-l)$$

*obey the Virasoro commutation relations*

$$[L_n^S, L_m^S] = (n-m)L_{n+m}^S - \delta_{n,-m} \frac{\dim \mathfrak{h}}{12} (n^3 - n).$$

*Proof.* Let us assume that  $h$ ,  $n$  and  $m$  are all odd for simplicity. We denote  $C^{ij} = \langle a^i, a^j \rangle$ . Moreover, assume  $m \geq 0$ .



Then we get

$$\begin{aligned}
(kh)^2[L_n^S, L_m^S] &= \sum_{\alpha < hn/2, \beta < hm/2, i, j} [a_{hn-\alpha}^i a_{\alpha}^{\bar{i}}, a_{hm-\beta}^j a_{\beta}^{\bar{j}}] \\
&= k \sum_{\alpha, \beta, i, j} a_{hn-\alpha}^i a_{\beta}^{\bar{j}} \alpha \delta_{\alpha, \beta-hm} \delta_{ij} + a_{\alpha}^{\bar{i}} a_{\beta}^{\bar{j}} (hn - \alpha) \delta_{hn-\alpha, \beta-hm} C^{ij} \\
&\quad + a_{hm-\beta}^j a_{\alpha}^{\bar{i}} (hn - \alpha) \delta_{hn-\alpha, -\beta} \delta_{ij} + a_{hm-\beta}^j a_{hn-\alpha}^i \alpha \delta_{\alpha, -\beta} C^{\bar{i}\bar{j}} \\
&= k \sum_{\alpha < hn/2, i} a_{hn-\alpha}^i a_{hm+\alpha}^{\bar{i}} \alpha \delta_{\alpha < -hm/2} + a_{\alpha}^{\bar{i}} a_{hn+hm-\alpha}^i (hn - \alpha) \delta_{\alpha > hn+hm/2} \\
&\quad + a_{hm+hn-\alpha}^i a_{\alpha}^{\bar{i}} (hn - \alpha) \delta_{\alpha < hn+hm/2} + a_{hm+\alpha}^{\bar{i}} a_{hn-\alpha}^i \alpha \delta_{\alpha > -hm/2}
\end{aligned}$$

We see that one may combine the terms to get

$$k \sum_{\alpha < hn/2, i} a_{hn-\alpha}^i a_{hm+\alpha}^{\bar{i}} \alpha + a_{hm+hn-\alpha}^i a_{\alpha}^{\bar{i}} (hn - \alpha).$$

Let's shift the index of summation in the first term:

$$k \sum_{\alpha < hn/2+hm, i} a_{hn+hm-\alpha}^i a_{\alpha}^{\bar{i}} (\alpha - hm) + k \sum_{\alpha < hn/2, i} a_{hm+hn-\alpha}^i a_{\alpha}^{\bar{i}} (hn - \alpha).$$

We want to make both sum to go up to  $\alpha = h(n+m)/2$ , so split off the necessary part in the first sum:

$$\begin{aligned}
&k \sum_{\alpha < h(n+m)/2, i} a_{hn+hm-\alpha}^i a_{\alpha}^{\bar{i}} (\alpha - hm) + k \sum_{h(n+m)/2 \leq \alpha < hn/2+hm, i} a_{hn+hm-\alpha}^i a_{\alpha}^{\bar{i}} (\alpha - hm) \\
&+ k \sum_{\alpha < hn/2, i} a_{hm+hn-\alpha}^i a_{\alpha}^{\bar{i}} (hn - \alpha).
\end{aligned}$$

The extra piece of the sum combines with the last term to finally give

$$\begin{aligned}
&k \sum_{\alpha < h(n+m)/2, i} a_{hn+hm-\alpha}^i a_{\alpha}^{\bar{i}} (hn - hm) + k \sum_i a_{h(n+m)/2}^i a_{h(n+m)/2}^{\bar{i}} h(n-m)/2 \\
&- k^2 \delta_{n, -m} \sum_{0 < \alpha < hm/2, i} \alpha (\alpha - hm).
\end{aligned}$$

The first two terms combine into  $(kh)^2(n-m)L_{n+m}$  (for  $n \neq -m$ ), while the last sum is

$$S := \sum_{0 < \alpha < hm/2} \alpha (\alpha - hm) d_{\alpha}.$$

Since  $d_\alpha$  depends on  $\alpha$  only modulo  $h$ , let us substitute  $\alpha = hb + l$ . Then we get

$$\begin{aligned}
S &= \sum_{-h/2 < l < h/2} d_l \sum_{-l/h < b < m/2 - l/h} (hb + l)(hb - hm + l) \\
&= \sum_{-h/2 < l < h/2} d_l \sum_{-l/h < b < m/2 - l/h} h^2 b^2 + l(l - hm) + 2hbl - h^2 mb \\
&= \sum_{0 < l < h/2} d_l l(l - hm) + \sum_{-h/2 < l < h/2} d_l \sum_{0 < b < m/2 - l/h} h^2 b^2 + l(l - hm) + 2hbl - h^2 mb \\
&= \sum_{0 < l < h/2} d_l l(l - hm) + \sum_{-h/2 < l < h/2} d_l \left( h^2 \frac{m^2 - 1}{24} m + l(l - hm) \frac{m - 1}{2} + \frac{m^2 - 1}{8} h(2l - hm) \right)
\end{aligned}$$

Since  $d_l = d_{-l}$ , only even powers of  $l$  survive in the last sum. Moreover,  $\sum_l d_l = \dim \mathfrak{h}$ . Therefore, we get

$$S = m \sum_{0 < l < h/2} d_l l(l - h) - h^2 \dim \mathfrak{h} \frac{m^2 - 1}{12} m.$$

The second term gives the usual Virasoro cocycle, while the first term can be absorbed into a redefinition of  $L_0^S$ .  $\square$

**3.2.2. Remark 1.** Note the unusual ordering in the Sugawara currents. We use this particular ordering as we will be interested in representations like  $\mathcal{O}(A/A_+)$ , which are not smooth, but instead dual to smooth ones (namely, the vacuum representation  $U\mathfrak{a}/U\mathfrak{a}_+$ ). This is also reflected in the minus sign in front of the central charge.

**3.2.3. Remark 2.** The Sugawara operators do not change if we multiply the bilinear form  $\langle, \rangle$  on  $\mathfrak{h}$  by a scalar.

**3.2.4.** We get an embedding of the Virasoro algebra into  $\widehat{U}_k(\widehat{\mathfrak{a}})$ . Now we want to see that the action of this copy of Virasoro coincides with the canonical Virasoro action on the Heisenberg.

**Proposition 10.** *The Sugawara currents  $L_n$  of [Proposition 9](#) have the following commutation relations with elements of  $\mathfrak{a}$ :*

$$[L_n, a_\alpha^i] = -\frac{\alpha}{h} a_{\alpha+hn}^i.$$

*Proof.* For simplicity, we again assume that both  $h$  and  $n$  are odd. Then the commutator is

$$\begin{aligned}
[L_n, a_\alpha^i] &= \frac{1}{kh} \sum_{\beta < nh/2, j} [a_{hn-\beta}^j a_\beta^{\bar{j}}, a_\alpha^i] \\
&= \frac{1}{h} \sum_{\beta < nh/2, j} (hn - \beta) \delta_{hn-\beta, -\alpha} C^{ji} a_\beta^{\bar{j}} + a_{hn-\beta}^j \beta \delta_{\beta, -\alpha} \delta_{ij} \\
&= -\frac{\alpha}{h} a_{\alpha+hn}^i.
\end{aligned}$$

$\square$

3.2.5. The last two propositions give an embedding

$$S : U_{c'}(\text{Vir}) \rightarrow \widehat{U}_{k,c}(\text{Vir} \ltimes \widehat{\mathfrak{a}})$$

given by

$$S(L_n) = L_n - L_n^S.$$

Indeed,

$$\begin{aligned} [S(L_n), S(L_m)] &= [L_n, L_m] + [L_n^S, L_m^S] - [L_n, L_m^S] - [L_n^S, L_m] \\ &= [L_n, L_m] - [L_n^S, L_m^S] \\ &= (n - m)S(L_{n+m}) + \delta_{n,-m} \frac{c + \dim \mathfrak{h}}{12} (n^3 - n). \end{aligned}$$

Therefore, the central charge is

$$c' = c + \dim \mathfrak{h}.$$

Furthermore, the image of  $S$  commutes with  $U_k(\widehat{\mathfrak{a}}) \subset \widehat{U}_{k,c}(\text{Vir} \ltimes \widehat{\mathfrak{a}})$ .

3.2.6. In this section we assume that  $X$  has genus 0. We have the following incarnation of the boson-fermion correspondence [Zh]:

**Proposition 11** (Frenkel–Kac–Segal). *Suppose  $G$  is simply-laced and  $\mathcal{L}$  is the basic ample line bundle on  $\text{Gr}_X$ . Then the restriction homomorphism  $H^0(LG_- \setminus LG, \mathcal{L}) \rightarrow H^0(A_- \setminus A, \mathcal{L})$  is an isomorphism. Moreover, it induces an isomorphism on the spaces of conformal blocks  $H^0(LG_- \setminus LG/LG_+, \mathcal{L}) \cong H^0(A_- \setminus A/A_+, \mathcal{L})$ .*

So, we have three isomorphic one-dimensional spaces:

$$H^0(\text{Bun}_G(X), \mathcal{L}) \cong H^0(\text{Gr}_X, \mathcal{L}) \cong H^0(A_- \setminus A/A_+, \mathcal{L}).$$

In particular, this implies that any section  $\sigma \in H^0(\widehat{\text{Gr}}_X, \mathcal{L})^{A_+}$  is invariant under the whole  $LG_+$ .

Any vector  $v \in H^0(X_0, T_X)$  acts by endomorphisms on  $H^0(\widehat{\text{Gr}}_X, \mathcal{L})$  and hence we have an action of  $S(v)$ . Note, that  $\widehat{\mathfrak{a}}$  acts on the right on  $H^0(\widehat{\text{Gr}}_X, \mathcal{L})$ , so the central charge is  $k = -1$ .

Since the operators  $S(v)$  commute with  $A_+$ , the action preserves the subspace of invariants  $H^0(\text{Gr}_X, \mathcal{L}) \subset H^0(\widehat{\text{Gr}}_X, \mathcal{L})$ . But, as we have just established,  $H^0(\text{Gr}_X, \mathcal{L})$  is one-dimensional. Since every element of  $H^0(X_0, T_X)$  is a commutator, every one-dimensional representation is trivial. We conclude:

**Proposition 12.** *Any vector field  $v \in H^0(X_0, T_X)$  preserves the section  $\sigma$ :  $\sigma(S(v)P) = S(v)\sigma(P)$ .*

This implies that the tau-function

$$\tau_P(g) = \frac{\sigma(g^{-1}P)}{g^{-1}\sigma(P)}$$

is invariant under the Sugawara currents on the right. But since  $g \in A$  and the Sugawara currents commute with  $A$ , it is also invariant under the Sugawara currents on the left.

Suppose  $a \in A_+$ . Then  $\tau_P(ag) = c(a, g)\tau_P(g)$ . Therefore,  $A_+$  acts on  $\tau_P(\cdot)g \in \mathcal{O}(A/A_+)$  via multiplication by a function in  $\mathcal{O}(A/A_+)$ . In contrast, elements in  $A/A_+$  act by translations.

Infinitesimally, it means that  $\mathfrak{a}/\mathfrak{a}_+$  acts on  $\tau_P(\cdot g)$  by vector fields, i.e. first-order differential operators.

Denote the projection  $p : \mathcal{A}_{stab} \rightarrow H^0(X_0, T_X)$ . Combining the invariance of the tau-function under the Sugawara currents and vectors  $a \in \mathcal{A}_{stab}$  we get

**Theorem 13.** *For any vector  $a \in \mathcal{A}_{stab}$  we have a second-order differential equation on the tau-function:*

$$(a - S(p(a)))\tau = 0.$$

These are the famous Virasoro constraints of two-dimensional quantum gravity [DVV]. For example, pick a global coordinate  $z^{-1}$  on  $\mathbf{P}^1$  vanishing to the first order at  $\infty$ . Let  $\{t_\alpha^i\}_{\alpha>0, i} \in \mathcal{O}(A/A_+)$  be the time coordinates, so that  $a_\alpha^i$  acts by  $\partial/\partial t_{-\alpha}^i$  for  $\alpha \leq 0$ , and it acts by  $(-k\alpha)t_\alpha^i$  for  $\alpha > 0$ .

Suppose  $p(g) = z^2 \frac{\partial}{\partial z}$  for some  $g \in \mathcal{A}_{stab}$ . Let's consider the case  $h = 2$ . Then the operator

$$S(z^2 \partial/\partial z) - g = L_1 - L_1^S - g = \frac{1}{2} \sum_{\alpha \geq 0, i} (\alpha + 2) t_{2+\alpha}^i \frac{\partial}{\partial t_\alpha^i} + \frac{1}{4} \sum_i (t_1^i)^2 - \sum_{i, \alpha} g_{i, \alpha} a_\alpha^i$$

is known as the string operator, where  $g_{i, \alpha}$  are some coefficients involved in the definition of  $g$ . Similarly,

$$S(z \partial/\partial z) - z^{-1}g = L_0 - L_0^S - z^{-1}g = \frac{1}{2} \sum_{\alpha > 0, i} \alpha t_\alpha^i \frac{\partial}{\partial t_\alpha^i} + \frac{1}{4} \sum_i \left( \frac{\partial}{\partial t_0^i} \right)^2 + \frac{d_1}{16} - \sum_{i, \alpha} g_{i, \alpha} a_{\alpha-2}^i,$$

which is related to the dilaton operator.

**3.2.7. Remark.** If  $X$  has a higher genus or  $G$  is not simply-laced,  $H^0(\text{Gr}_X, \mathcal{L})$  is no longer one-dimensional, so we have a collection of tau-functions parametrized by a basis  $\{\sigma_i\}$  of  $H^0(\text{Gr}_X, \mathcal{L})$ :

$$\tau_P^{ij}(g) = \frac{\sigma_i(g^{-1}P)}{g^{-1}\sigma_j(P)}.$$

The Sugawara currents give a representation of  $H^0(X_0, T_X)$  on the space of tau-functions by acting on the index  $i$ . Therefore, in general we do not expect constraints on a single tau-function, but instead relations between different ones.

### 3.3. Differential side.

**3.3.1.** The Grassmannian description of the Drinfeld-Sokolov hierarchy is a geometric reinterpretation of the original Drinfeld-Sokolov phase space, which parametrizes certain objects called affineopers. In this section we briefly recall what these are and explain what it means for an affine oper to be string. The reader is referred to [BZF1] for a complete description.

Consider two groups  $G$  and  $K \subset G$  and let  $\mathbf{O} \subset \mathfrak{g}/\mathfrak{k}$  be a  $K$ -orbit for the adjoint action. Suppose  $\mathcal{E} \rightarrow C$  is a  $G$ -bundle and  $\mathcal{E}_K \rightarrow C$  is a reduction to a subgroup  $K \subset G$ . We denote by  $\mathcal{A}_P$  and  $\mathcal{A}_{P_K}$  the corresponding Atiyah bundles.

**Definition.** We say that  $\mathcal{E}_K$  has relative position  $\mathbf{O}$  with respect to a connection  $\nabla : T_C \rightarrow \mathcal{A}_\mathcal{E}$  if the image of  $\nabla$  is contained inside of  $\mathcal{A}_{\mathcal{E}_K}$  up to elements of  $\mathbf{O} \times_K \mathcal{E}_K$ .

If  $\mathbf{O} = 0$  it simply means that  $\nabla$  is induced from a connection on  $P_K$  and thus we say that  $\mathcal{E}_K$  is *flat* with respect to  $\nabla$ .

3.3.2. The *Iwahori subgroup*  $LG^+ \subset LG_+$  is defined as the subgroup of loops which take values in a fixed Borel subgroup  $B \subset G$  at the closed point.

Let  $p_{-1} = \sum_{i=0}^n f_i \in L\mathfrak{g}_+$  be the principal nilpotent element and  $\mathbf{O}^{aff}$  the  $\mathbf{C}^\times$ -span of the  $LG^+$ -orbit of  $p_{-1}$ .

$D_t = \text{Spec } \mathbf{C}[[t]]$  will denote a formal disk; note, that it is independent of the formal disk with a local coordinate  $z$  in the definition of  $LG$ .

**Definition.** An *affine oper* on  $D_t$  is an  $LG$ -torsor  $\mathcal{E} \rightarrow D_t$  together with a connection  $\nabla$ , a flat reduction  $\mathcal{E}_{LG_-}$  to  $LG_-$  and a reduction  $\mathcal{E}_{LG^+}$  to  $LG^+$  in relative position  $\mathbf{O}^{aff}$  with respect to  $\nabla$ .

Alternatively, an affine oper is a collection of the following data:

- A  $G$ -torsor  $P \rightarrow X \times D_t$ .
- A  $B$ -reduction  $P_B$  along  $\infty \times D_t$ .
- A relative connection  $\nabla_{oper}$  on  $X_0 \times D_t$  in the  $D_t$  direction, with an asymptotic condition that  $\nabla$  extends to a connection at  $\infty$  preserving the  $B$ -reduction up to elements in  $\mathbf{O}^{aff}$ .

Although the definition may seem complicated at first, it can be made quite explicit in certain cases. For example, if  $X$  has genus 0,  $G = GL_n \mathbf{C}$  and the underlying  $G$ -bundle on  $X$  is trivial, this is simply an  $n$ -th order differential operator on  $D_t$  with symbol 1.

The moduli space of affineopers on  $D_t$  will be denoted by  $\text{AOp}_X(D_t)$ .

An important observation of Drinfeld and Sokolov is that affineopers have a unique  $A_+$ -reduction, where  $A$  is the principal Heisenberg, which has relative position  $\mathbf{C}^\times \cdot p_{-1} \in \mathfrak{a}/\mathfrak{a}_+$  with respect to the oper connection  $\nabla_{oper}$ .

Let  $A_{-1} \subset A/A_+$  be the subgroup exponentiating the vector field  $p_{-1}$ . Then for any point  $P \in \text{Gr}_X$  we have a family of  $LG$ -bundles on  $A_{-1}$  together with reductions to  $LG_-$  and  $A_+$ . Moreover, the action gives a connection  $\nabla$ .

**Proposition 14** (Ben-Zvi–Frenkel). *Thus defined map  $\text{Gr}_X \rightarrow \text{AOp}_X(A_{-1})$  is an isomorphism.*

Suppose now that  $P \in \text{Gr}_X$  is a string point. Then there is a connection  $\nabla_{string}$  on  $P \rightarrow X_0$ , which preserves the reduction to  $A$  near infinity. Hence, the oper connection  $\nabla_{oper}$  is flat with respect to  $\nabla_{string}$ , i.e. together they form an absolute flat connection on  $X_0 \times A_{-1}$ . Combining this with [Theorem 7](#) we get a converse statement.

**Theorem 15.** *An affine oper  $(P, P_B, \nabla)$  is string iff the relative oper connection  $\nabla_{oper}$  on  $X_0 \times A_{-1}$  extends to an absolute flat connection preserving the  $A$ -reduction near infinity.*

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